

A UNIVERSAL DEFORMATION FORMULA FOR CONNES-MOSCOVICI'S HOPF ALGEBRA WITHOUT ANY PROJECTIVE STRUCTURE

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ABSTRACT. We construct a universal deformation formula for Connes-Moscovici's Hopf algebra without any projective structure using Fedosov's quantization of symplectic diffeomorphisms.

1. INTRODUCTION

In the study of index theory of a transverse elliptic differential operator of a codimension one foliation, Connes and Moscovici discovered a Hopf algebra \mathcal{H}_1 which governs the local symmetry in computing the Chern character. In this paper, we study deformation theory of this Hopf algebra. In particular, we prove that the Hopf algebra \mathcal{H}_1 has a universal deformation formula.

In [4], inspired from Rankin-Cohen brackets on modular forms, Connes and Moscovici constructed a universal deformation formula of the Hopf algebra \mathcal{H}_1 with a projective structure. By a universal deformation formula of a Hopf algebra A , we mean an element $R \in A[[\hbar]] \otimes A[[\hbar]]$ satisfying

$$(\Delta \otimes 1)R(R \otimes 1) = (1 \otimes \Delta)R(1 \otimes R), \text{ and } \epsilon \otimes 1(R) = 1 \otimes 1 = 1 \otimes \epsilon(R).$$

In [1], we together with Bieliavsky provided a geometric interpretation of a projective structure in the case of a codimension one foliation. And as a result, we (with Bieliavsky) obtained a geometric way to reconstruct Connes-Moscovici's universal deformation formula. Furthermore, a new and interesting result we proved in [1][Prop. 6.1] is that even without a projective structure, the first Rankin-Cohen bracket

$$RC_1 = S(X) \otimes Y + Y \otimes X \in \mathcal{H}_1 \otimes \mathcal{H}_1$$

is a noncommutative Poisson structure, i.e. RC_1 is a Hochschild cocycle and $(1 \otimes \Delta)RC_1(1 \otimes RC_1) - (\Delta \otimes 1)RC_1(RC_1 \otimes 1)$ is a Hochschild coboundary. This leads to a question whether \mathcal{H}_1 has a universal deformation formula without a projective structure.

In this article, we give a positive answer to the above question and introduce a geometric construction of a universal deformation formula of \mathcal{H}_1 without a projective structure. The idea of this construction goes back to Fedosov [6] in his study of deformation quantization of a symplectic diffeomorphism. Fedosov developed in [6] a systematic way to quantize a symplectic diffeomorphism to an endomorphism of the quantum algebra no matter whether it preserves or not the chosen symplectic connection. Fedosov also observed that composition of quantized symplectic diffeomorphisms does not preserve associativity. Instead, it satisfies a weaker associativity property—associative up to an inner endomorphism. This picture can be explained using the language of “gerbes and stacks” as [2]. In any case, Fedosov's construction does give rise to a deformation quantization of the groupoid algebra associated to a pseudogroup action on a symplectic manifold.

In this paper, we apply this idea to the special case that the symplectic manifold is $\mathbb{R} \times \mathbb{R}^+$ and the Poisson structure is $\partial_x \wedge \partial_y$, with x the coordinate on \mathbb{R} and y the coordinate on \mathbb{R}^+ .

We consider symplectic diffeomorphisms on $\mathbb{R} \times \mathbb{R}^+$ of the form

$$\gamma : (x, y) \rightarrow \left(\gamma(x), \frac{y}{\gamma'(x)} \right),$$

where γ is a local diffeomorphism on \mathbb{R} . In this case, Fedosov's construction of quantization of symplectic diffeomorphism can be computed explicitly. In particular, we are able to prove that the resulting star product on the groupoid algebra $C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$ can be expressed by

$$f\alpha \star g\beta = m(R(f\alpha \otimes g\beta)),$$

where m is the multiplication map on $C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$, and R is an element in $\mathcal{H}_1[[\hbar]] \otimes \mathcal{H}_1[[\hbar]]$. An important property is that the \mathcal{H}_1 action on the collection of all $C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$ for all pseudogroup Γ is fully injective because this action is equivalent to the action used by Connes and Moscovici to define \mathcal{H}_1 . With this observation, we can derive all the property of R as a universal deformation formula from the corresponding properties about the star product on $C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$.

The notion of universal deformation formula of a Hopf algebra is closely related to the solution of quantum Yang-Baxter equation. We hope that our construction will shed a light on the study of deformation theory of the Hopf algebra \mathcal{H}_1 and also codimension one foliations.

This article is organized as follows. We review in section 2 Fedosov's theory of deformation quantization of symplectic diffeomorphisms. We provide a detailed proof of the fact that this defines a deformation of the groupoid algebra $C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$. In section 3, we prove the main theorem of this paper that \mathcal{H}_1 has a universal deformation formula using Fedosov's theory reviewed in section 2. In section 4, we compute explicitly our universal deformation formula up to \hbar^2 . We observe that when the \mathcal{H}_1 action is projective, the universal deformation formula obtained in this paper does not agree with the one introduced by Connes and Moscovici in [4]. Instead, our lower order term computation suggests that in the case of a projective action, these two universal deformation formulas should be related by an isomorphism expressed by elements in $\mathcal{H}_1[[\hbar]]$ and the projective structure Ω . In the appendix we discuss the associativity of the Eholzer product on modular forms, which was used by Connes and Moscovici in constructing their Rankin-Cohen deformation.

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2. QUANTIZATION OF SYMPLECTIC DIFFEOMORPHISMS

In this section, we briefly recall Fedosov's construction of quantization of a symplectic diffeomorphism. And we use this idea to define a deformation of a groupoid algebra coming from a pseudogroup action on a symplectic manifold. We learned this construction from A. Gorokhovsky, R. Nest, and B. Tsygan.

In Fedosov's approach to deformation quantization of a symplectic manifold (M, ω) , a flat connection D (also called Fedosov connection) on the Weyl algebra bundle \mathcal{W} plays an essential role. The quantum algebra is identified with the space of flat sections \mathcal{W}_D of \mathcal{W} . A question arises when one wants to quantize a symplectic diffeomorphism. Because a symplectic diffeomorphism may not preserve D , the canonical lifting of a symplectic diffeomorphism to the Weyl algebra bundle \mathcal{W} may not act on the quantum algebra \mathcal{W}_D . How can we quantize a symplectic diffeomorphism in this case? Fedosov in [6] studied this question. The answer he came up with

fits well with the language of “stack of algebras”. In the following we briefly review Fedosov’s results [6][Section 4].

A symplectic diffeomorphism $\gamma : M \rightarrow M$ naturally acts on the cotangent bundle $\gamma : T^*M \rightarrow T^*M$. Therefore γ lifts to an endomorphism on the Weyl algebra bundle $\gamma : \mathcal{W} \rightarrow \mathcal{W}$. It is easy to check that if $\gamma(D) := \gamma \circ D \circ \gamma^{-1} = D$, then γ defines an algebra endomorphism on the quantum algebra $\mathcal{W}_D = \ker(D)$, which is called a quantization of the symplectic diffeomorphism γ . We with Bieliavsky in [1] used this idea to construct a universal deformation formula of \mathcal{H}_1 with a projective structure.

The quantization of γ when $\gamma(D) \neq D$ is more involved. Fedosov in [6][Sec. 4] purposed the following construction of quantization. We start with extending the standard Weyl algebra W to W^+ ,

- (1) an element u of W^+ can be written as

$$u = \sum_{2k+l \geq 0} \hbar^k a_{k,i_1,\dots,i_l} y^{i_1} \cdots y^{i_l}$$

where (y^1, \dots, y^{2n}) are coordinates on the standard symplectic vector space (V, ω) . In the above sum, we allow k to be negative.

- (2) There are a finite number of terms with a given total degree $2k + l \geq 0$.

We remark that the Moyal product extends to a well define product on W^+ . And we consider the corresponding extension \mathcal{W}^+ of the Weyl algebra bundle \mathcal{W} associated to W^+ . Write $\gamma(D) = D + i/\hbar[\Delta_\gamma, \cdot]$ with Δ_γ a smooth section of the Weyl algebra bundle \mathcal{W} . We remark that in principle, we can change Δ_γ by any section in the center of \mathcal{W} . But when we fix the expression of $D = d + i/\hbar[r, \cdot]$, Δ_γ has a canonical choice $\gamma(r) - r$. In the following of this paper, we will always work with this choice of Δ_γ . We consider the following equation

$$(1) \quad DU_\gamma = -\frac{i}{\hbar} \Delta_\gamma \circ U_\gamma,$$

where U_γ is an invertible section of \mathcal{W}^+ . Fedosov [6][Thm 4.3] proved that equation (1) always has solutions. In general, solutions to equation (1) are not unique. But the following induction procedure

$$U_{n+1} = 1 + \delta^{-1}\{(D + \delta)U_n + (i/\hbar)\Delta_\gamma \circ U_n\}, \quad U_0 = 1$$

uniquely determines an invertible solution to Equation (1). By this induction, we see that U is a solution to the following equation

$$(2) \quad U = 1 + \delta^{-1}\{(D + \delta)U + i/\hbar\Delta_\gamma \circ U\},$$

which actually has a unique solution because $\delta^{-1}\{(D + \delta)U + i/\hbar\Delta_\gamma \circ U\}$ raises the total degree of U by 1. We will always work with this solution in this paper. By Equation (1), U_α^{-1} satisfies the following equation,

$$\begin{aligned} DU_\alpha^{-1} &= -U_\alpha^{-1} \circ DU_\alpha \circ U_\alpha^{-1} \\ &= \frac{i}{\hbar} U_\alpha^{-1} \circ \Delta_\alpha. \end{aligned}$$

Using Equation (2), it is not difficult to check that U_α^{-1} is the unique solution to the following equation,

$$(3) \quad V = 1 + \delta^{-1}\{(D + \delta)V - i/\hbar V \circ \Delta_\alpha\},$$

which can be constructed by same induction as above.

We have the following lemma on Δ_α .

Lemma 2.1. (1) $\Delta_\alpha + \alpha(\Delta_\beta) = \Delta_{\alpha\beta}$,

$$(2) \quad \alpha(\Delta_{\alpha^{-1}}) = -\Delta_{\alpha}.$$

Proof. (1) $D + i/\hbar[\Delta_{\alpha\beta}, \cdot] = \alpha\beta(D) = \alpha(\beta(D)) = \alpha(D + i/\hbar[\Delta_{\beta}, \cdot]) = \alpha(D) + i/\hbar[\alpha(\Delta_{\beta}), \cdot] = D + i/\hbar[\Delta_{\alpha}, \cdot] + i/\hbar[\alpha(\Delta_{\beta}), \cdot] = D + i/\hbar[\Delta_{\alpha} + \alpha(\Delta_{\beta}), \cdot]$. Therefore by the defining property of $\Delta_{\alpha\beta}$ and its uniqueness, we conclude that $\Delta_{\alpha\beta} = \Delta_{\alpha} + \alpha(\Delta_{\beta})$.

(2) Corollary of (1). \square

We will need the following properties of U_{α} later in our construction.

Proposition 2.2. *The assignment $\alpha \mapsto U_{\alpha}$ satisfies the following properties,*

- (1) $D(\alpha(U_{\beta})) = -i/\hbar(\Delta_{\alpha\beta}) \circ \alpha(U_{\beta}) + i/\hbar\alpha(U_{\beta}) \circ \Delta_{\alpha};$
- (2) $\alpha(U_{\alpha^{-1}}) = U_{\alpha}^{-1}.$

Proof. (1) We can use Equation (2) to prove a stronger statement. We compute

$$\begin{aligned} \alpha(U_{\beta}) &= \alpha(1 + \delta^{-1}\{(D + \delta)U_{\beta} + i/\hbar\Delta_{\beta} \circ U_{\beta}\}) \\ &= 1 + \delta^{-1}\{(\alpha(D) + \delta)\alpha(U_{\beta}) + i/\hbar\alpha(\Delta_{\beta}) \circ \alpha(U_{\beta})\} \\ &= 1 + \delta^{-1}\{(D + \delta)\alpha(U_{\beta}) + i/\hbar[\Delta_{\alpha}, \alpha(U_{\beta})] + i/\hbar\alpha(\Delta_{\beta}) \circ \alpha(U_{\beta})\} \\ &= 1 + \delta^{-1}\{(D + \delta)\alpha(U_{\beta}) + i/\hbar(\Delta_{\alpha} + \alpha(\Delta_{\beta})) \circ \alpha(U_{\beta}) - i/\hbar\alpha(U_{\beta}) \circ \Delta_{\alpha}\}. \end{aligned}$$

By applying Lemma 2.1 to the last line, we conclude that $\alpha(U_{\beta})$ is the unique solution to the following equation

$$(4) \quad \alpha(U_{\beta}) = 1 + \delta^{-1}\{(D + \delta)\alpha(U_{\beta}) + i/\hbar\Delta_{\alpha\beta} \circ \alpha(U_{\beta}) - i/\hbar\alpha(U_{\beta}) \circ \Delta_{\alpha}\}.$$

We remark that solution to Equation (4) is unique because $\delta^{-1}\{(D + \delta)\alpha(U_{\beta}) + i/\hbar\Delta_{\alpha\beta} \circ \alpha(U_{\beta}) - i/\hbar\alpha(U_{\beta}) \circ \Delta_{\alpha}\}$ raises the total degree of $\alpha(U_{\beta})$ by 1. Taking δ on both sides of the above equation, we obtain the first identity of this Proposition.

(2) Setting $\beta = \alpha^{-1}$ in Equation (4), we have that $\alpha(U_{\alpha^{-1}}) = 1 + \delta^{-1}\{(D + \delta)\alpha(U_{\alpha^{-1}}) + i/\hbar(\Delta_{id} \circ \alpha(U_{\alpha^{-1}}) - i/\hbar\alpha(U_{\alpha^{-1}}) \circ \Delta_{\alpha})\} = 1 + \delta^{-1}\{(D + \delta)\alpha(U_{\alpha^{-1}}) - i/\hbar\alpha(U_{\alpha^{-1}}) \circ \Delta_{\alpha}\}$, which is same as the defining equation for U_{α}^{-1} . By the uniqueness of the solution to the above equation, we have that $\alpha(U_{\alpha^{-1}}) = U_{\alpha}^{-1}$. \square

Fedosov [6] defined quantization of a symplectic diffeomorphism γ on (M, ω) as

$$\hat{\gamma}(a) = Ad_{U_{\gamma}^{-1}}(\gamma(a)) = U_{\gamma}^{-1} \circ \gamma(a) \circ U_{\gamma},$$

which is an algebra endomorphism of the quantum algebra \mathcal{W}_D .

The “defect” of this quantization is that the associativity of composition fails, i.e.

$$\hat{\alpha}\hat{\beta} \neq \widehat{\alpha\beta}.$$

Instead, Fedosov proves the following property of the associator $v_{\alpha,\beta} := U_{\alpha}^{-1} \circ \alpha(U_{\beta}^{-1}) \circ \alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})$.

Proposition 2.3. *The associator $v_{\alpha,\beta}$ is a flat section of \mathcal{W} and*

$$\hat{\alpha}\hat{\beta}(\widehat{\alpha\beta})^{-1} = Ad_{v_{\alpha,\beta}}.$$

Proof. Using $DU_\alpha^{-1} = i/\hbar U_\alpha^{-1} \circ \Delta_\alpha$, we have

$$\begin{aligned}
Dv_{\alpha,\beta} &= D\left(U_\alpha^{-1} \circ \alpha(U_\beta^{-1}) \circ \alpha(\beta(U_{(\alpha\beta)^{-1}}^{-1}))\right) \\
&= D(U_\alpha^{-1}) \circ \alpha(U_\beta^{-1}) \circ \alpha(\beta(U_{(\alpha\beta)^{-1}}^{-1})) \\
&\quad + U_\alpha^{-1} \circ D(\alpha(U_\beta^{-1})) \circ \alpha\beta(U_{(\alpha\beta)^{-1}}^{-1}) \\
&\quad + U_\alpha^{-1} \circ \alpha(U_\beta^{-1}) \circ D(\alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})) \\
&= \frac{i}{\hbar} U_\alpha^{-1} \circ \Delta_\alpha \circ \alpha(U_\beta^{-1}) \circ \alpha\beta(U_{(\alpha\beta)^{-1}}^{-1}) \\
&\quad - \frac{i}{\hbar} U_\alpha^{-1} \circ (\Delta_\alpha \circ \alpha(U_\beta^{-1}) - \alpha(U_\beta^{-1}) \circ \Delta_{\alpha\beta}) \circ \alpha\beta(U_{(\alpha\beta)^{-1}}^{-1}) \\
&\quad - \frac{i}{\hbar} U_\alpha^{-1} \circ \alpha(U_\beta^{-1}) \circ \Delta_{\alpha\beta} \circ \alpha\beta(U_{(\alpha\beta)^{-1}}^{-1}) \\
&= 0.
\end{aligned}$$

We remark that in the above formula we have used the property $D(\alpha(U_\beta^{-1})) = i/\hbar \alpha(U_\beta^{-1}) \circ \Delta_{\alpha\beta} - i/\hbar \Delta_\alpha \circ \alpha(U_\beta)^{-1}$. Therefore, by [6][Lemma 4.2], we conclude that v is a flat section of \mathcal{W} .

The property of the associator is a straight forward computation. \square

We use $f \mapsto \hat{f}$ to represent the isomorphism between $C^\infty(M)$ and the quantum algebra \mathcal{W}_D . We know that $\alpha(\hat{g})$ is a flat section of the connection $\alpha(D)$. Therefore, $\alpha(\hat{g})$ satisfies the following equation

$$D(\alpha(\hat{g})) = -\frac{i}{\hbar} [\Delta_\alpha, \alpha(\hat{g})].$$

Then $U_\alpha^{-1} \circ \alpha(\hat{g}) \circ U_\alpha$ satisfies the following equation

$$\begin{aligned}
&D(U_\alpha^{-1} \circ \alpha(\hat{g}) \circ U_\alpha) \\
&= D(U_\alpha^{-1}) \circ \alpha(\hat{g}) \circ U_\alpha + U_\alpha^{-1} \circ D(\alpha(\hat{g})) \circ U_\alpha + U_\alpha^{-1} \circ \alpha(\hat{g}) \circ D(U_\alpha) \\
&= \frac{i}{\hbar} U_\alpha^{-1} \circ \Delta_\alpha \circ \alpha(\hat{g}) \circ U_\alpha - \frac{i}{\hbar} U_\alpha^{-1} \circ (\Delta_\alpha \circ \alpha(\hat{g}) - \alpha(\hat{g}) \circ \Delta_\alpha) \circ U_\alpha - \frac{i}{\hbar} U_\alpha^{-1} \circ \alpha(\hat{g}) \circ \Delta_\alpha \circ U_\alpha \\
&= 0
\end{aligned}$$

In the following, we apply the above idea to quantize the groupoid algebra of a pseudogroup Γ on a symplectic manifold M . We define the following product on $C_c^\infty(M) \rtimes \Gamma[[\hbar]]$,

$$\begin{aligned}
f\alpha \star g\beta &:= \left(\hat{f} \circ U_\alpha^{-1} \circ \alpha(\hat{g}) \circ U_\alpha \circ v_{\alpha,\beta} \right) \Big|_{y=0} \alpha\beta \\
&= \left(\hat{f} \circ U_\alpha^{-1} \circ \alpha(\hat{g}) \circ \alpha(U_\beta^{-1}) \circ \alpha\beta(U_{(\alpha\beta)^{-1}}^{-1}) \right) \Big|_{y=0} \alpha\beta,
\end{aligned}$$

where y 's are coordinates along the fiber direction of T^*M . We remark that because \hat{f} , $U_\alpha^{-1} \circ \alpha(\hat{g}) \circ U_\alpha$, and $v_{\alpha,\beta}$ are all flat with respect to the connection D , the product $\hat{f} \circ U_\alpha^{-1} \circ \alpha(\hat{g}) \circ \alpha(U_\beta^{-1}) \circ \alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})$ is also flat with respect to the connection D .

We check the associativity of \star on $C_c^\infty(M) \rtimes \Gamma[[\hbar]]$. We compute

$$\begin{aligned}
& (f\alpha \star g\beta) \star h\gamma \\
&= \left(\hat{f} \circ U_\alpha^{-1} \circ \alpha(\hat{g}) \circ \alpha(U_\beta^{-1}) \circ \alpha\beta(U_{(\alpha\beta)^{-1}}^{-1}) \right) \big|_{y=0} \alpha\beta \star h\gamma \\
&= \left(\hat{f} \circ U_\alpha^{-1} \circ \alpha(\hat{g}) \circ \alpha(U_\beta^{-1}) \circ \alpha\beta(U_{(\alpha\beta)^{-1}}^{-1}) \circ U_{\alpha\beta}^{-1} \circ \alpha\beta(\hat{h}) \circ \right. \\
&\quad \left. \circ \alpha\beta(U_\gamma^{-1}) \circ \alpha\beta\gamma(U_{(\alpha\beta\gamma)^{-1}}^{-1}) \right) \big|_{y=0} \alpha\beta\gamma \\
&= \left(\hat{f} \circ U_\alpha^{-1} \circ \alpha(\hat{g}) \circ \alpha(U_\beta^{-1}) \circ \alpha\beta(\hat{h}) \circ \alpha\beta(U_\gamma^{-1}) \circ \alpha\beta\gamma(U_{(\alpha\beta\gamma)^{-1}}^{-1}) \right) \big|_{y=0} \alpha\beta\gamma,
\end{aligned}$$

where we have used $\alpha\beta(U_{(\alpha\beta)^{-1}}^{-1}) = U_{\alpha\beta}$, and

$$\begin{aligned}
& f\alpha \star (g\beta \star h\gamma) \\
&= f\alpha \star \left(\hat{g} \circ U_\beta^{-1} \circ \beta(\hat{h}) \circ \beta(U_\gamma^{-1}) \circ \beta\gamma(U_{(\beta\gamma)^{-1}}^{-1}) \right) \big|_{y=0} \beta\gamma \\
&= \left(\hat{f} \circ U_\alpha^{-1} \circ \alpha(\hat{g} \circ U_\beta^{-1} \circ \beta(\hat{h}) \circ \beta(U_\gamma^{-1}) \circ \beta\gamma(U_{(\beta\gamma)^{-1}}^{-1})) \circ \alpha(U_{\beta\gamma}^{-1}) \circ \alpha\beta\gamma(U_{(\alpha\beta\gamma)^{-1}}^{-1}) \right) \big|_{y=0} \alpha\beta\gamma \\
&= \left(\hat{f} \circ U_\alpha \circ \alpha(\hat{g}) \circ \alpha(U_\beta^{-1}) \circ \alpha\beta(\hat{h}) \circ \alpha\beta(U_\gamma^{-1}) \circ \alpha\beta\gamma(U_{(\alpha\beta\gamma)^{-1}}^{-1}) \right) \big|_{y=0} \alpha\beta\gamma,
\end{aligned}$$

where we have used $\beta\gamma(U_{(\beta\gamma)^{-1}}^{-1}) = U_{\beta\gamma}$.

We conclude that \star defines an associative product on the algebra $C_c^\infty(M) \rtimes \Gamma[[\hbar]]$.

3. A UNIVERSAL DEFORMATION FORMULA

In this section, we apply the construction described in the previous section to construct a universal deformation formula of Connes-Moscovici's Hopf algebra \mathcal{H}_1 . We start with briefly recalling the definition of \mathcal{H}_1 .

We consider the defining representation of \mathcal{H}_1 on $C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$. Let (x, y) with $y > 0$ be coordinates on $\mathbb{R} \times \mathbb{R}^+$. Define $\gamma : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R} \times \mathbb{R}^+$ by

$$\gamma(x, y) = \left(\gamma(x), \frac{y}{\gamma'(x)} \right).$$

We remark that the above expression of γ action does not agree with the formulas in [3], but the two actions are isomorphic under the transformation $y \mapsto 1/y$.

Consider $X = 1/y \partial_x$, and $Y = -y \partial_y$ acting on $C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$ as follows

$$X(f\alpha) = \frac{1}{y} f_x \alpha, \quad Y(f\alpha) = -y f_y \alpha.$$

We have the following observation

$$\alpha(Y(\alpha^{-1}(f))) = \alpha \left(Y(f(\alpha(x), \frac{y}{\alpha'(x)})) \right) = \alpha \left(-\frac{y}{\alpha'(x)} f_y(\alpha(x), \frac{y}{\alpha'(x)}) \right) = -y f_y(x, y) = Y(f),$$

and

$$\begin{aligned}
\alpha(X(\alpha^{-1}(f))) &= \alpha \left(X(f(\alpha(x), \frac{y}{\alpha'(x)})) \right) = \alpha \left(\frac{1}{y} \alpha'(x) f_x(\alpha(x), \frac{y}{\alpha'(x)}) - \frac{\alpha''(x)}{(\alpha'(x))^2} f_y(\alpha(x), \frac{y}{\alpha'(x)}) \right) \\
&= \frac{1}{y} f_x + \frac{\alpha^{-1''}}{\alpha^{-1'}} f_y = Xf - \frac{\log(\alpha^{-1'})'}{y} Yf = (X - \delta_1(\alpha)Y)f
\end{aligned}$$

where $\delta_1(f\alpha) = \log(\alpha^{-1'})'/y f$.

We compute $\delta_2(\alpha) = X(\delta_1(\alpha)) = \frac{1}{y} \partial_x (\log(\alpha^{-1'})'/y) = \frac{1}{y^2} \frac{\alpha^{-1'''} \alpha^{-1'} - (\alpha^{-1''})^2}{(\alpha^{-1'})^2}$, and define $\delta_n(\alpha) = X(\delta_{n-1}(\alpha))$ by induction for $n \geq 2$.

On $\mathbb{R} \times \mathbb{R}^+$, we consider the Poisson structure $\partial_x \wedge \partial_y$, which can be expressed by $-X \otimes Y + Y \otimes X$. Our main goal is to use the method reviewed in the previous section to construct a star product on $C_c^\infty(M) \rtimes \Gamma[[\hbar]]$. We prove that this star product as a bilinear operator actually can be expressed by an element R of $\mathcal{H}_1 \otimes \mathcal{H}_1[[\hbar]]$. The associativity of the star product is equivalent to the property that R is a universal deformation formula. We start with fixing a symplectic connection ∇ on the cotangent bundle of $\mathbb{R} \times \mathbb{R}^+$, which was introduced in [1][Sec. 3]

$$\nabla_{\partial_x} \partial_x = 0, \nabla_{\partial_x} \partial_y = \frac{1}{2y} \partial_x, \nabla_{\partial_y} \partial_x = \frac{1}{2y} \partial_x, \nabla_{\partial_y} \partial_y = -\frac{1}{2y} \partial_y.$$

Using X and Y , we can express the above connection by

$$\nabla_X X = 0, \nabla_X Y = -\frac{1}{2} X, \nabla_Y X = \frac{1}{2} X, \nabla_Y Y = -\frac{1}{2} Y.$$

We compute $\alpha(\nabla)$ by $\alpha \nabla \alpha^{-1}$.

$$\begin{aligned} \alpha(\nabla)_{\partial_x} \partial_x &= -\frac{\alpha^{-1'''} \alpha^{-1'} - \frac{3}{2} (\alpha^{-1''})^2}{(\alpha^{-1'})^2} y \partial_y, & \alpha(\nabla)_{\partial_x} \partial_y &= \frac{1}{2y} \partial_x, \\ \alpha(\nabla)_{\partial_y} \partial_x &= \frac{1}{2y} \partial_x, & \alpha(\nabla)_{\partial_y} \partial_y &= -\frac{1}{2y} \partial_y, \end{aligned}$$

and

$$\alpha(\nabla)_X X = \delta'_2(\alpha) Y, \alpha(\nabla)_X Y = -\frac{1}{2} X, \alpha(\nabla)_Y X = \frac{1}{2} X, \alpha(\nabla)_Y Y = -\frac{1}{2} Y,$$

where $\delta'_2 = \delta_2 - \frac{1}{2} \delta_1^2$. We observe that both ∇ and $\alpha(\nabla)$ are flat and torsion free.

We consider the lifting of ∇ and $\alpha(\nabla)$ onto the Weyl algebra bundle. Use u, v to denote the generators along the fiber direction of the Weyl algebra bundle \mathcal{W} . We have for any section a of \mathcal{W} ,

$$\begin{aligned} Da &= da - dx \frac{\partial a}{\partial u} - dy \frac{\partial a}{\partial v} + \frac{i}{\hbar} \left[\frac{1}{2y} v^2 dx + \frac{1}{2y} 2uv dy, a \right], \\ \alpha(D)a &= da - dx \frac{\partial a}{\partial u} - dy \frac{\partial a}{\partial v} + \frac{i}{\hbar} \left[(y^3 \delta'_2(\alpha) u^2 + \frac{1}{2y} v^2) dx + \frac{1}{2y} 2uv dy, a \right]. \end{aligned}$$

Therefore, using the notation of the previous section, we have that $\Delta_\alpha = y^3 \delta'_2(\alpha) u^2 dx$.

In the following, we solve the expression for $\hat{f}, \alpha(\hat{g}), U_\alpha^{-1}, \alpha(U_\beta^{-1}), \alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})$.

3.1. \hat{f} . The section \hat{f} of \mathcal{W} is a unique solution of

$$D\hat{f} = 0, \quad \hat{f}|_{u=v=0} = f.$$

Write $\hat{f} = \sum_{mn} f_{mn} u^m v^n$. Then the above equation is expressed as

$$\begin{aligned} & \sum_{mn} \left(dx \partial_x f_{mn} u^m v^n + dy \partial_y f_{mn} u^m v^n - dx f_{mn} m u^{m-1} v^n - dy f_{mn} u^m n v^{n-1} \right. \\ & \quad \left. + dx \frac{1}{2y} \frac{-1}{2} 2v f_{mn} m u^{m-1} v^n + dy \frac{1}{2y} \frac{1}{2} (2v f_{mn} u^m n v^{n-1} - 2u f_{mn} m u^{m-1} v^n) \right) \\ &= \sum_{mn} dx \left(\partial_x f_{mn} - (m+1) f_{m+1n} - \frac{1}{2y} (m+1) f_{m+1n-1} \right) u^m v^n \\ & \quad + \sum_{mn} dy \left(\partial_y f_{mn} - (n+1) f_{mn+1} + \frac{1}{2y} (n-m) f_{mn} \right) u^m v^n. \end{aligned}$$

Therefore, we have that \hat{f} is the unique solution to the following family of equations

$$(5) \quad \begin{aligned} \partial_x f_{mn} - (m+1)f_{m+1n} - \frac{1}{2y}(m+1)f_{m+1n-1} &= 0 \\ \partial_y f_{mn} - (n+1)f_{mn+1} + \frac{1}{2y}(n-m)f_{mn} &= 0 \end{aligned}$$

with $f_{00} = f$.

Solving Equation (5), we have that

$$\begin{aligned} f_{mn} &= \frac{1}{m!n!} \left(\partial_y - \frac{m+1-n}{2y} \right) \cdots \left(\partial_y - \frac{m}{2y} \right) \partial_x^m f \\ &= \frac{(-1)^n}{m!} y^{m-n} X^m \left(Y + \frac{m+n-1}{2} \right) \cdots \left(Y + \frac{m}{2} \right) (f). \end{aligned}$$

3.2. $\alpha(\hat{g})$. We know that $\alpha(\hat{g})$ is the unique solution to the following equation

$$D(\alpha(\hat{g})) = -\frac{i}{\hbar} [\Delta_\alpha, \alpha(\hat{g})],$$

with $\alpha(\hat{g})|_{u=v=0} = \alpha(g)$. We remind that $\Delta_\alpha = y^3 \delta'_2(\alpha) u^2 dx$.

Similar to \hat{f} , $\alpha(\hat{g})$ satisfies the following equation

$$\begin{aligned} 0 &= \sum_{mn} dx \left(\partial_x \alpha(g)_{mn} - (m+1)\alpha(g)_{m+1n} \right. \\ &\quad \left. - \frac{1}{2y}(m+1)\alpha(g)_{m+1n-1} + y^3 \delta'_2(n+1)\alpha(g)_{m-1n+1} \right) u^m v^n \\ &\quad + \sum_{mn} dy \left(\partial_y \alpha(g)_{mn} - (n+1)\alpha(g)_{mn+1} + \frac{1}{2y}(n-m)\alpha(g)_{mn} \right) u^m v^n \end{aligned}$$

with $\alpha(g)_{00} = \alpha(g)$. Therefore, $\alpha(\hat{g})$ is the unique solution to the following family of equations

$$(6) \quad \begin{aligned} 0 &= \partial_x \alpha(g)_{mn} - (m+1)\alpha(g)_{m+1n} - \frac{1}{2y}(m+1)\alpha(g)_{m+1n-1} + y^3 \delta'_2(n+1)\alpha(g)_{m-1n+1} \\ 0 &= \partial_y \alpha(g)_{mn} - (n+1)\alpha(g)_{mn+1} + \frac{1}{2y}(n-m)\alpha(g)_{mn} \end{aligned}$$

with $\alpha(g)_{00} = \alpha(g)$.

By the second equation of Equations (6), we have

$$\alpha(g)_{m,n} = \frac{1}{n} \left(\partial_y + \frac{n-1-m}{2y} \right) \alpha(g)_{m,n-1} = \cdots = \frac{1}{n!} \left(\partial_y + \frac{n-1-m}{2y} \right) \cdots \left(\partial_y + \frac{-m}{2y} \right) \alpha(g)_{m,0}.$$

Setting $n = 0$ in the first equation of Equations (6), we obtain

$$\begin{aligned} \alpha(g)_{m+1,0} &= \frac{1}{m+1} (\partial_x \alpha(g)_{m,0} + y^3 \delta'_2 \alpha(g)_{m-1,1}) \\ &= \frac{1}{m+1} \left(\partial_x \alpha(g)_{m,0} + y^3 \delta'_2 \left(\partial_y + \frac{1-m}{2y} \right) \alpha(g)_{m-1,0} \right). \end{aligned}$$

By induction, we can solve the above equation as

$$\alpha(g)_{mn} = \frac{(-1)^n y^{m-n}}{m!n!} A_m \left(Y + \frac{n+m-1}{2} \right) \cdots \left(Y + \frac{m}{2} \right) (\alpha(g)),$$

where $A_m \in \mathcal{H}_1$ is defined inductively by

$$A_{m+1} = X A_m - m \delta'_2 \left(Y - \frac{m-1}{2} \right) A_{m-1}, \quad A_0 = 1.$$

3.3. U_α^{-1} . We compute U_α^{-1} using the equation

$$DU_\alpha^{-1} = U_\alpha^{-1} \circ \frac{i}{\hbar} \Delta_\alpha$$

with $\Delta_\alpha = \delta'_2(\alpha) y^3 u^2 dx$.

Write $U_\alpha^{-1} = \sum_{mn} u_{mn}^\alpha u^m v^n$, where u_{mn}^α takes values in $C_c^\infty(M)[\hbar^{-1}, \hbar]$. And u_{mn}^α satisfies the following family of equations

$$(7) \quad \begin{aligned} 0 &= \partial_x u_{mn}^\alpha - (m+1)u_{m+1n}^\alpha - \frac{1}{2y}(m+1)u_{m+1n-1}^\alpha - \frac{i}{\hbar} y^3 \delta'_2 u_{m-2n}^\alpha \\ &\quad + y^3 \delta'_2 (n+1)u_{m-1n+1}^\alpha + \frac{i\hbar}{4} y^3 \delta'_2 (n+2)(n+1)u_{mn+2}^\alpha, \\ 0 &= \partial_y u_{mn}^\alpha - (n+1)u_{mn+1}^\alpha + \frac{1}{2y}(n-m)u_{mn}^\alpha \end{aligned}$$

with $u_{0,0} = 1$.

The second equation of Equations (7) implies that

$$u_{mn}^\alpha = \frac{1}{n} (\partial_y + \frac{n-1-m}{2y}) u_{mn-1}^\alpha = \cdots = \frac{1}{n!} (\partial_y + \frac{n-1-m}{2y}) \cdots (\partial_y + \frac{-m}{2y}) u_{m,0}^\alpha.$$

We use the $n=0$ version of the first equation of Equation (7) to solve u_{m0} .

$$\begin{aligned} u_{m+10}^\alpha &= \frac{1}{m+1} \left(\partial_x u_{m0}^\alpha - \frac{i}{\hbar} y^3 \delta'_2 u_{m-20}^\alpha + y^3 \delta'_2 (\partial_y - \frac{m-1}{2y}) u_{m-1}^\alpha \right. \\ &\quad \left. + \frac{i\hbar}{4} y^3 \delta'_2 (\partial_y - \frac{m-1}{2y}) (\partial_y - \frac{m}{2y}) u_{m0}^\alpha \right) \end{aligned}$$

By induction, we have the following expression of u ,

$$u_{mn}^\alpha = \frac{(-1)^n y^{m-n}}{m!n!} (Y + \frac{n-m-1}{2}) \cdots (Y - \frac{m}{2}) A_m 1,$$

where A_m is defined by

$$A_{m+1} = \left(X + \frac{i\hbar}{4} \delta'_2 (Y - \frac{m-1}{2}) (Y - \frac{m}{2}) \right) A_m - \delta'_2 (Y - \frac{m-1}{2}) A_{m-1} - \frac{i}{\hbar} \delta'_2 A_{m-2},$$

with $A_0 = 1$.

Remark 3.1. We need to prove that the above obtained solution $\tilde{U}_\alpha^{-1} = \sum u_{mn}^\alpha u^m v^n$ is the unique solution to the defining Equation (3) of U_α^{-1} , which implies that the above $\tilde{U}_\alpha^{-1} = U_\alpha^{-1}$.

Using the above expression for u_{mn}^α , by induction we can prove that the negative power of \hbar contained in u_{mn}^α is less than or equal to $[m/3]$ ($[\mu]$ means the Gauss integer function.). This shows that once $m+n > 0$, the smallest degree term contained in $u_{mn}^\alpha u^m v^n$ has degree greater than or equal to 1. Therefore the degree 0 term of the solution $\tilde{U}_\alpha^{-1} = \sum u_{mn}^\alpha u^m v^n$ is equal to 1. Accordingly, we compute using that $D\tilde{U}_\alpha^{-1} - \tilde{U}_\alpha^{-1} \circ i/\hbar \Delta_\alpha = 0$,

$$\begin{aligned} \tilde{U}_\alpha^{-1} &= \delta \delta^{-1} \tilde{U}_\alpha^{-1} + \delta^{-1} \delta \tilde{U}_\alpha^{-1} + 1 = 1 + \delta^{-1} \delta \tilde{U}_\alpha^{-1} \\ &= 1 + \delta^{-1} (\delta \tilde{U}_\alpha^{-1} + D\tilde{U}_\alpha^{-1} - \tilde{U}_\alpha^{-1} \circ i/\hbar \Delta_\alpha) \\ &= 1 + \delta^{-1} \{ (D + \delta) \tilde{U}_\alpha^{-1} - \tilde{U}_\alpha^{-1} \circ i/\hbar \Delta_\alpha \}. \end{aligned}$$

This remark applies also to the following solutions $\alpha(U_\beta^{-1})$ and $\alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})$.

3.4. $\alpha(U_\beta^{-1})$ **and** $\alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})$. By Proposition 2.2, we know that $\alpha(U_\beta)$ satisfies the equation

$$D(\alpha(U_\beta)) = -\frac{i}{\hbar}(\Delta_{\alpha\beta} \circ \alpha(U_\beta) - \alpha(U_\beta) \circ \Delta_\alpha).$$

Accordingly, $\alpha(U_\beta^{-1}) = (\alpha(U_\beta))^{-1}$ satisfies

$$\begin{aligned} D(\alpha(U_\beta^{-1})) &= -(\alpha(U_\beta))^{-1} \circ D(\alpha(U_\beta)) \circ (\alpha(U_\beta))^{-1} \\ &= -(\alpha(U_\beta))^{-1} \circ \left(-\frac{i}{\hbar}\Delta_{\alpha\beta} \circ \alpha(U_\beta) + \frac{i}{\hbar}\alpha(U_\beta) \circ \Delta_\alpha \right) \circ (\alpha(U_\beta))^{-1} \\ &= \frac{i}{\hbar}(\alpha(U_\beta^{-1}) \circ \Delta_{\alpha\beta} - \Delta_\alpha \circ \alpha(U_\beta^{-1})) \end{aligned}$$

If we write $\alpha(U_\beta^{-1}) = \sum u_{mn}^{\alpha,\beta} u^m v^n$, then we have

$$\begin{aligned} 0 &= \partial_x u_{mn}^{\alpha,\beta} - (m+1)u_{m+1n}^{\alpha,\beta} - \frac{1}{2y}(m+1)u_{m+1n-1}^{\alpha,\beta} - \frac{i}{\hbar}y^3\alpha(\delta'_2(\beta))u_{m-2n}^{\alpha,\beta} \\ (8) \quad &+ y^3(n+1)(2\delta'_2(\alpha) + \alpha(\delta'_2(\beta)))u_{m-1n+1}^{\alpha,\beta} + \frac{i\hbar}{4}y^3\alpha(\delta'_2(\beta))u_{mn+2}^{\alpha,\beta} \\ 0 &= \partial_y u_{mn}^{\alpha,\beta} - (n+1)u_{mn+1}^{\alpha,\beta} + \frac{1}{2y}(n-m)u_{mn}^{\alpha,\beta}. \end{aligned}$$

We can solve Equation (8) of $u_{mn}^{\alpha,\beta}$ as follows

$$u_{m,n}^{\alpha,\beta} = \frac{(-1)^n y^{m-n}}{m!n!} \left(Y + \frac{n-m-1}{2}\right) \cdots \left(Y - \frac{m}{2}\right) A_m 1,$$

where $A_m \in \mathcal{H}_1$ is defined inductively

$$\begin{aligned} A_{m+1} &= \left(X + \frac{i\hbar}{4}\alpha(\delta'_2(\beta))(Y - \frac{m-1}{2})(Y - \frac{m}{2}) \right) A_m \\ &\quad - (2\delta'_2(\alpha) + \alpha(\delta'_2(\beta)))(Y - \frac{m-1}{2})A_{m-1} - \frac{i}{\hbar}\alpha(\delta'_2(\beta))A_{m-2}, \quad A_0 = 1. \end{aligned}$$

We know that $\alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})$ is equal to $U_{\alpha\beta}$, which satisfies

$$DU_{\alpha\beta} = -\frac{i}{\hbar}\Delta_{\alpha\beta} \circ U_{\alpha\beta}.$$

We can solve $\alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})$ as $U_{\alpha\beta}$. Write $\alpha\beta(U_{(\alpha\beta)^{-1}}^{-1}) = U_{\alpha\beta} = \sum v_{mn}^{\alpha\beta} u^m v^n$. Then

$$v_{mn}^{\alpha\beta} = \frac{(-1)^n y^{m-n}}{m!n!} \left(Y + \frac{n-m-1}{2}\right) \cdots \left(Y - \frac{m}{2}\right) A_m 1,$$

where $A_m \in \mathcal{H}_1$ is defined by

$$A_{m+1} = \left(X - \frac{i\hbar}{4}\delta'_2(\alpha\beta)(Y - \frac{m-1}{2})(Y - \frac{m}{2}) \right) A_m + \delta'_2(\alpha\beta)(Y - \frac{m-1}{2})A_{m-1} + \frac{i}{\hbar}\delta'_2(\alpha\beta)A_{m-2},$$

with $A_0 = 1$.

We notice that terms surviving in the product $\hat{f} \circ U_\alpha^{-1} \circ \alpha(\hat{g}) \circ \alpha(U_\beta^{-1}) \circ \alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})|_{u=v=0}$ are sums of terms of the following form

$$C_{m_1, \dots, m_5; n_1, \dots, n_5} = f_{m_1 n_1} u_{m_2 n_2}^\alpha \alpha(g)_{m_3 n_3} u_{m_4 n_4}^{\alpha, \beta} v_{m_5 n_5}^{\alpha, \beta} u^{m_1} v^{n_1} \circ \cdots \circ u^{m_5} v^{n_5} |_{u=v=0}$$

with $m_1 + \cdots + m_5 = n_1 + \cdots + n_5$.

Theorem 3.2. *There exists an element $R \in \mathcal{H}_1 \otimes \mathcal{H}_1[[\hbar]]$ such that the star product on $C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$ can be expressed by $f\alpha \star g\beta = m(R(f\alpha \otimes g\beta))$, where $m : C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma \otimes C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma \rightarrow C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$ is the multiplication map. Furthermore, R is a universal deformation formula of \mathcal{H}_1 .*

Proof. We start by rewriting $u_{mn}^\alpha, u_{mn}^{\alpha,\beta}, v_{mn}^{\alpha\beta}$, and $\alpha(\hat{g})$.

- (1) As X and Y vanishes on 1, u_{mn}^α can be written as y^{m-n} times a sum of terms like powers of X and Y acting on powers of $\delta'_2(\alpha)$. If we rewrite $\delta'_2(\alpha)$ as $\delta_2 - 1/2\delta_1^2$, we can write a term of powers of X and Y acting on powers of $\delta'_2(\alpha)$ into a sum of products like $\delta_{i_1}^{j_1}(\alpha) \cdots \delta_{i_p}^{j_p}(\alpha)$. We point out that as there are $1/\hbar$ in the induction formula of A_m , u_{mn}^α may contain terms with negative powers of \hbar . However, by induction on the negative power of \hbar , we can easily prove that the negative power of \hbar is no more than $[m/3]$. Therefore, we can write u_{mn}^α as $\hbar^{-[m/3]} y^{m-n} \mu_{mn}(\delta_1(\alpha), \delta_2(\alpha), \dots)$, where μ_{mn} is a polynomial of variables $\hbar, \delta_1, \delta_2, \dots$ independent of α .
- (2) Analogous to the above analysis, we have that $u_{m,n}^{\alpha,\beta}$ can be written as y^{m-n} times a sum of terms like powers of X and Y acting on products of powers of $\delta'_2(\alpha)$ and $\alpha(\delta'_2(\beta))$. When X and Y act on $\delta'_2(\alpha)$, we can write the resulting terms into polynomials of $\delta_1(\alpha), \dots, \delta_p(\alpha), \dots$. To compute X and Y action on $\alpha(\delta'_2(\beta))$, we look at the following properties of X and Y , for any function f

$$\begin{aligned} X(\alpha(f)) &= \alpha(\alpha^{-1}(X(\alpha(f)))) = \alpha(X(f) - \delta_1(\alpha^{-1})Y(f)) \\ &= \alpha(X(f)) - \alpha(\delta_1(\alpha^{-1}))\alpha(Y(f)) = \alpha(X(f)) + \delta_1(\alpha)\alpha(Y(f)) \\ Y(\alpha(f)) &= \alpha(\alpha^{-1}(Y(\alpha(f)))) = \alpha(Y(f)), \end{aligned}$$

where we have used the commutation relation between X, Y and α . This implies that powers of X, Y acting on $\alpha(\delta'_2(\beta))$ gives a sum of terms like $\sigma(\delta_1(\alpha), \dots)\alpha(\xi(\delta_1(\beta), \dots))$ with ν, ξ polynomials in $\mathcal{H}_1[[\hbar]]$ independent of α, β . We summarize that $u_{mn}^{\alpha,\beta}$ can be written as

$$\hbar^{-[m/3]} y^{m-n} \sum_i \nu_{mn}^i(\delta_1(\alpha), \dots) \alpha(\xi_{mn}^i(\delta_1(\beta), \dots))$$

where ν^i, ξ^i are polynomials of $\hbar, \delta_1, \delta_2, \dots$, independent of α, β .

- (3) Similar to u_{mn}^α , $v_{mn}^{\alpha\beta}$ can be written as a sum of terms like powers of X, Y acting on $\delta'_2(\alpha\beta)$. For our purpose, we need to rewrite $\delta'_2(\alpha\beta)$ by a sum of $\delta'_2(\alpha) + \alpha(\delta'_2(\beta))$. Therefore the situation is similar to $u_{mn}^{\alpha,\beta}$. We can write $v_{mn}^{\alpha\beta}$ as

$$\hbar^{-[m/3]} y^{m-n} \sum_j \eta_{mn}^j(\delta_1(\alpha), \dots) \alpha(\lambda_{mn}^j(\delta_1(\beta), \dots))$$

with $\eta_{mn}^j, \lambda_{mn}^j$ polynomials independent of α, β .

- (4) From the inductive relations, we see that $\alpha(\hat{g})_{mn}$ can be written as a sum of terms like product of two parts. One part is powers of X and Y acting on $\delta'_2(\alpha)$, the other is powers of X and Y acting on $\alpha(g)$. We can write the part involving $\delta'_2(\alpha)$ as polynomials of $\delta_1(\alpha), \delta_2(\alpha), \dots$, the part with $\alpha(g)$ like the above $\alpha(\delta'_2(\beta))$ as a sum of terms like

$$\varphi(\delta_1(\alpha), \dots) \alpha(\phi(X, Y)(g)).$$

Therefore, we can write $\alpha(\hat{g})_{mn}$ as

$$\sum y^{m-n} \rho_{mn}^i(\delta_1(\alpha), \dots) \alpha(\psi_{mn}^i(X, Y)(g)).$$

Summarizing the above consideration, we can write the term $C_{m_1, \dots, m_5; n_1, \dots, n_5}$ as

$$c_{m_1, \dots, m_5; n_1, \dots, n_5} \cdot \hbar^{m_1+m_2-\lfloor \frac{m_2}{3} \rfloor + m_3+m_4-\lfloor \frac{m_4}{3} \rfloor + m_5-\lfloor \frac{m_5}{3} \rfloor} \sum_{i,j,k} \tau_{m_1 n_1}(X, Y)(f) \mu_{m_2 n_2}(\delta_1(\alpha), \dots) \\ \rho_{m_3 n_3}^i(\delta_1(\alpha), \dots) \alpha(\psi_{m_3 n_3}^i(X, Y)(g)) \\ v_{m_4 n_4}^j(\delta_1(\alpha), \dots) \alpha(\xi_{m_4 n_4}^j(\delta_1(\beta), \dots)) \eta_{m_5 n_5}^k(\delta_1(\alpha), \dots) \alpha(\lambda_{m_5 n_5}^k(\delta_1(\beta), \dots)),$$

where $c_{m_1, \dots, m_5; n_1, \dots, n_5}$ is a constant. And $\hat{f} \circ U_\alpha^{-1} \circ \alpha(\hat{g}) \circ \alpha(U_\beta^{-1}) \circ \alpha\beta(U_{(\alpha\beta)^{-1}}^{-1})|_{u=v=0} \alpha\beta$ can be written

$$\sum_{\substack{m_1, \dots, m_5; n_1, \dots, n_5 \\ m_1 + \dots + m_5 = n_1 + \dots + n_5}} c_{m_1, \dots, m_5; n_1, \dots, n_5} \hbar^{m_1+m_2-\lfloor \frac{m_2}{3} \rfloor + m_3+m_4-\lfloor \frac{m_4}{3} \rfloor + m_5-\lfloor \frac{m_5}{3} \rfloor} \\ \sum_{i,j,k} \tau_{m_1 n_1}(X, Y)(f) \mu_{m_2 n_2}(\delta_1(\alpha), \dots) \rho_{m_3 n_3}^i(\delta_1(\alpha), \dots) \alpha(\psi_{m_3 n_3}^i(X, Y)(g)) \\ v_{m_4 n_4}^j(\delta_1(\alpha), \dots) \alpha(\xi_{m_4 n_4}^j(\delta_1(\beta), \dots)) \eta_{m_5 n_5}^k(\delta_1(\alpha), \dots) \alpha(\lambda_{m_5 n_5}^k(\delta_1(\beta), \dots)) \alpha\beta \\ = \sum_{m_1, \dots, m_5; n_1, \dots, n_5} c_{m_1, \dots, m_5; n_1, \dots, n_5} \hbar^{m_1+m_2-\lfloor \frac{m_2}{3} \rfloor + m_3+m_4-\lfloor \frac{m_4}{3} \rfloor + m_5-\lfloor \frac{m_5}{3} \rfloor} \\ \sum_{i,j,k} \mu_{m_2 n_2}(\delta_1(\alpha), \dots) \rho_{m_3 n_3}^i(\delta_1(\alpha), \dots) v_{m_4 n_4}^j(\delta_1(\alpha), \dots) \eta_{m_5 n_5}^k(\delta_1(\alpha), \dots) \tau_{m_1 n_1}(X, Y)(f) \\ \alpha(\xi_{m_4 n_4}^j(\delta_1(\beta), \dots) \lambda_{m_5 n_5}^k(\delta_1(\beta), \dots) \psi_{m_3 n_3}^i(X, Y)(g)) \alpha\beta \\ = \sum_{m_1, \dots, m_5; n_1, \dots, n_5} c_{m_1, \dots, m_5; n_1, \dots, n_5} \hbar^{m_1+m_2-\lfloor \frac{m_2}{3} \rfloor + m_3+m_4-\lfloor \frac{m_4}{3} \rfloor + m_5-\lfloor \frac{m_5}{3} \rfloor} \\ \sum_{i,j,k} \mu_{m_2 n_2}(\delta_1(\alpha), \dots) \rho_{m_3 n_3}^i(\delta_1(\alpha), \dots) v_{m_4 n_4}^j(\delta_1(\alpha), \dots) \eta_{m_5 n_5}^k(\delta_1(\alpha), \dots) \tau_{m_1 n_1}(X, Y)(f) \alpha \\ \xi_{m_4 n_4}^j(\delta_1(\beta), \dots) \lambda_{m_5 n_5}^k(\delta_1(\beta), \dots) \psi_{m_3 n_3}^i(X, Y)(g) \beta$$

Define $R_{m_1, \dots, m_5; n_1, \dots, n_5} \in \mathcal{H}_1[\hbar] \otimes \mathcal{H}_1[\hbar]$ as

$$c_{m_1, \dots, m_5; n_1, \dots, n_5} \sum_{i,j,k} \mu_{m_2 n_2}(\delta_1, \dots) \rho_{m_3 n_3}^i(\delta_1, \dots) \nu_{m_4 n_4}^j(\delta_1, \dots) \eta_{m_5 n_5}^k(\delta_1, \dots) \tau_{m_1 n_1}(X, Y) \\ \otimes \xi_{m_4 n_4}^j(\delta_1, \dots) \lambda_{m_5 n_5}^k(\delta_1, \dots) \psi_{m_3 n_3}^i(X, Y)$$

Furthermore, we define

$$R = \sum_{m_1 + \dots + m_5 = n_1 + \dots + n_5} \hbar^{m_1+m_2-\lfloor \frac{m_2}{3} \rfloor + m_3+m_4-\lfloor \frac{m_4}{3} \rfloor + m_5-\lfloor \frac{m_5}{3} \rfloor} R_{m_1, \dots, m_5; n_1, \dots, n_5}.$$

We conclude that $R \in \mathcal{H}_1[[\hbar]] \otimes \mathcal{H}_1[[\hbar]]$ satisfies

$$f \alpha \star g \beta = m(R(f \alpha \otimes g \beta)).$$

To check that R is a universal deformation formula, we need to check that

- (1) $(\Delta \otimes 1)R(R \otimes 1) = (1 \otimes \Delta)R(1 \otimes R)$. This is from the fact that \star is associative and the \mathcal{H}_1 action on the collection of $C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$ over all pseudogroups Γ is fully injective.
- (2) $(\epsilon \otimes 1)R = 1 \otimes 1 = (1 \otimes \epsilon)R$. This is equivalent to show that 1 is a unit respect to the \star product on $C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma$.

When f is 1 and α is identity, we have that $\hat{f} = 1$, $U_{\text{id}}^{-1} = 1$, and $U_{\beta}^{-1} \circ \beta(U_{\beta^{-1}}^{-1}) = 1$. This implies that $1 \star g\beta = g\beta$.

When g is 1 and β is identity, we have that $\alpha(\hat{1}) = 1$, $U_{\beta}^{-1} = 1$ and $U_{\alpha}^{-1} \circ \alpha(U_{\alpha^{-1}}^{-1}) = 1$. Therefore, $f\alpha \star 1 = f\alpha$.

□

From the computation in the next section, we know that $R \in 1 \otimes 1 + \hbar R'$, where R' is an element in $\mathcal{H}_1[[\hbar]] \otimes \mathcal{H}_1[[\hbar]]$. Therefore, R is invertible with $R^{-1} = 1 + \sum_i (-1)^i (\hbar R')^i$. By the property of universal deformation formula, we can introduce a new Hopf algebra structure on $\mathcal{H}_1[[\hbar]]$ by twisting the coproduct Δ by

$$\tilde{\Delta}(a) = R^{-1} \Delta(a) R,$$

and the antipode by

$$\tilde{S}(a) = v^{-1} S(a) v,$$

with $v = m(S \otimes 1)(R)$.

4. FORMULAS OF LOWER ORDER TERMS

We must say that the formula for R constructed in the previous section Theorem 3.2 could be very complicated. We do not know an easy way to write it down explicitly. We will provide in this section the computation of R up to the second order of \hbar .

It is not difficult to check that if $m_1 + m_2 - [m_2/3] + m_3 + m_4 - [m_4/3] + m_5 - [m_5/3] = 0$ for nonnegative integers m_i , $i = 1, \dots, 5$, then $m_1 = \dots = m_5 = 0$. Therefore, the \hbar^0 component of the \star product $f\alpha \star g\beta$ is equal to $f\alpha(g)\alpha\beta$. This implies that $R_{0,\dots,0;0,\dots,0} = 1$ and $R = 1 \otimes 1 + O(\hbar)$.

We consider $R_{m_1,\dots,m_5;n_1,\dots,n_5}$ with $m_1 + m_2 - [m_2/3] + m_3 + m_4 - [m_4/3] + m_5 - [m_5/3] = 1$. It is not difficult to find that this implies one of m_i , $i = 1, \dots, 5$ takes value 1, and all others vanish. We also check that $u_{11}^{\alpha} = u_{01}^{\alpha} = u_{10}^{\alpha} = u_{11}^{\alpha,\beta} = u_{10}^{\alpha,\beta} = u_{01}^{\alpha,\beta} = v_{11}^{\alpha\beta} = v_{10}^{\alpha\beta} = v_{01}^{\alpha\beta} = 0$. Therefore, $R_{1,0,\dots,0;0,0,1,0,0}$ and $R_{0,0,1,0,0;1,0,\dots,0}$ are the only nonzero terms among all $R_{m_1,\dots,m_5;n_1,\dots,n_5}$ with

$$m_1 + m_2 - [m_2/3] + m_3 + m_4 - [m_4/3] + m_5 - [m_5/3] = 1.$$

We compute $R_{1,0,\dots,0;0,0,1,0,0} = \frac{i\hbar}{2} X \otimes Y$, and $R_{0,0,1,0,0;1,0,\dots,0} = -\frac{i\hbar}{2} (\delta_1 Y \otimes Y + Y \otimes X)$. Therefore, the \hbar component of R is

$$\frac{-i\hbar}{2} (-X \otimes Y + \delta_1 Y \otimes Y + Y \otimes X) = \frac{-i\hbar}{2} (S(X) \otimes Y + Y \otimes X).$$

We consider $R_{m_1,\dots,m_5;n_1,\dots,n_5}$ with $m_1 + m_2 - [m_2/3] + m_3 + m_4 - [m_4/3] + m_5 - [m_5/3] = 2$. There are three classes of possibilities. i) one of m_2, m_4, m_5 is equal to 3; ii) one of m_i ($i = 1, \dots, 5$) is equal to 2, iii) two of m_i ($i = 1, \dots, 5$) are both equal to 1. We notice that $u_{1.}^{\alpha} = u_{2.}^{\alpha} = u_{1.}^{\alpha,\beta} = u_{2.}^{\alpha,\beta} = v_{1.}^{\alpha\beta} = v_{2.}^{\alpha\beta} = 0$. This implies that terms contributing in $m_1 + m_2 - [m_2/3] + m_3 + m_4 - [m_4/3] + m_5 - [m_5/3] = 2$ are from following three groups.

- (1) $R_{0,3,0,0,0;1,0,2,0,0}$, $R_{0,3,0,0,0;2,0,1,0,0}$, $R_{0,0,0,3,0;1,0,2,0,0}$,
 $R_{0,0,0,3,0;2,0,1,0,0}$, $R_{0,0,0,0,3;1,0,2,0,0}$, $R_{0,0,0,0,3;2,0,1,0,0}$;
- (2) $R_{0,3,0,0,0;3,0,\dots,0}$, $R_{0,3,0,0,0;0,0,3,0,0}$, $R_{0,0,0,3,0;3,0,\dots,0}$,
 $R_{0,0,0,3,0;0,0,3,0,0}$, $R_{0,0,0,0,3;3,0,\dots,0}$, $R_{0,0,0,0,3;0,0,3,0,0}$;
- (3) $R_{2,0,\dots,0;0,0,2,0,0}$, $R_{0,0,2,0,0;2,0,0,0,0}$, $R_{1,0,1,0,0;1,0,1,0,0}$.

We compute the above terms separately.

- $R_{0,3,0,0,0;1,0,2,0,0} = (\frac{-i\hbar}{2})^3 \frac{1}{2} \frac{-i}{\hbar} \delta_2' Y \otimes (Y + \frac{1}{2}) Y = -(\frac{-i\hbar}{2})^2 \frac{1}{4} \delta_2' Y \otimes (Y + \frac{1}{2}) Y$.
- $R_{0,3,0,0,0;2,0,1,0,0} = (\frac{-i\hbar}{2})^3 \frac{1}{2!} \frac{i}{\hbar} \delta_2' (Y + \frac{1}{2}) Y \otimes Y = (\frac{-i\hbar}{2})^2 \frac{1}{4} \delta_2' (Y + \frac{1}{2}) Y \otimes Y$.
- $R_{0,0,0,3,0;1,0,2,0,0} = -(\frac{-i\hbar}{2})^3 \frac{1}{2} \frac{i}{\hbar} Y \otimes \delta_2' (Y + \frac{1}{2}) Y = -(\frac{-i\hbar}{2})^2 \frac{1}{4} Y \otimes \delta_2' (Y + \frac{1}{2}) Y$.

- $R_{0,0,0,3,0;2,0,1,0,0} = -(\frac{-i\hbar}{2})^3 \frac{1}{2} \frac{i}{\hbar} (Y + \frac{1}{2})Y \otimes \delta'_2 Y = -(\frac{-i\hbar}{2})^2 \frac{1}{4} (Y + \frac{1}{2})Y \otimes \delta'_2 Y.$
- $R_{0,0,0,0,3;1,0,2,0,0} = (\frac{-i\hbar}{2})^3 \frac{1}{2} \frac{i}{\hbar} [\delta'_2 Y \otimes (Y + \frac{1}{2})Y + Y \otimes \delta'_2 (Y + \frac{1}{2})Y] = (\frac{-i\hbar}{2})^2 \frac{1}{4} [\delta'_2 Y \otimes (Y + \frac{1}{2})Y + Y \otimes \delta'_2 (Y + \frac{1}{2})Y].$
- $R_{0,0,0,0,3;2,0,1,0,0} = (\frac{-i\hbar}{2})^3 \frac{1}{2} \frac{i}{\hbar} [\delta'_2 (Y + \frac{1}{2})Y \otimes Y + (Y + \frac{1}{2})Y \otimes \delta'_2 Y] = (\frac{-i\hbar}{2})^2 \frac{1}{4} [\delta'_2 (Y + \frac{1}{2})Y \otimes Y + (Y + \frac{1}{2})Y \otimes \delta'_2 Y].$
- $R_{0,3,0,0,0;3,0,0,0,0} = -(\frac{-i\hbar}{2})^3 \frac{i}{6\hbar} \delta'_2 (Y + 1)(Y + \frac{1}{2})Y \otimes 1 = -(\frac{-i\hbar}{2})^2 \frac{1}{12} \delta'_2 (Y + 1)(Y + \frac{1}{2})Y \otimes 1.$
- $R_{0,3,0,0,0;0,0,3,0,0} = (\frac{-i\hbar}{2})^3 \frac{i}{6\hbar} \delta'_2 \otimes (Y + 1)(Y + \frac{1}{2})Y = (\frac{-i\hbar}{2})^2 \frac{1}{12} \delta'_2 \otimes (Y + 1)(Y + \frac{1}{2})Y.$
- $R_{0,0,0,3,0;3,0,0,0,0} = -(\frac{-i\hbar}{2})^3 \frac{i}{6\hbar} (Y + 1)(Y + \frac{1}{2})Y \otimes \delta'_2 = -(\frac{-i\hbar}{2})^2 \frac{1}{12} (Y + 1)(Y + \frac{1}{2})Y \otimes \delta'_2.$
- $R_{0,0,0,3,0;0,0,3,0,0} = -(\frac{-i\hbar}{2})^3 \frac{i}{6\hbar} 1 \otimes \delta'_2 (Y + 1)(Y + \frac{1}{2})Y = -(\frac{-i\hbar}{2})^2 \frac{1}{12} 1 \otimes \delta'_2 (Y + 1)(Y + \frac{1}{2})Y.$
- $R_{0,0,0,0,3;3,0,0,0,0} = (\frac{-i\hbar}{2})^3 \frac{i}{6\hbar} [\delta'_2 (Y + 1)(Y + \frac{1}{2})Y \otimes 1 + (Y + 1)(Y + \frac{1}{2})Y \otimes \delta'_2] = (\frac{-i\hbar}{2})^2 \frac{1}{12} [\delta'_2 (Y + 1)(Y + \frac{1}{2})Y \otimes 1 + (Y + 1)(Y + \frac{1}{2})Y \otimes \delta'_2].$
- $R_{0,0,0,0,3;0,0,3,0,0} = (\frac{-i\hbar}{2})^3 \frac{i}{6\hbar} [\delta'_2 \otimes (Y + 1)(Y + \frac{1}{2})Y + 1 \otimes \delta'_2 (Y + 1)(Y + \frac{1}{2})Y] = (\frac{-i\hbar}{2})^2 \frac{1}{12} [\delta'_2 \otimes (Y + 1)(Y + \frac{1}{2})Y + 1 \otimes \delta'_2 (Y + 1)(Y + \frac{1}{2})Y].$
- $R_{2,0,\dots,0;0,0,2,0,0} = (\frac{-i\hbar}{2})^2 \frac{1}{2} X^2 \otimes (Y + \frac{1}{2})Y.$
- $R_{0,0,2,0,0;2,0,\dots,0,0} = (\frac{-i\hbar}{2})^2 \frac{1}{2} \left((Y + \frac{1}{2})Y \otimes X^2 + \delta'_2 (Y + \frac{1}{2})Y \otimes Y + 2\delta_1 (Y + \frac{1}{2})Y \otimes XY + \delta'_2 (Y + \frac{1}{2})Y \otimes X + \delta_1^2 (Y + \frac{1}{2})Y \otimes Y^2 + \frac{1}{2}\delta_1^2 (Y + \frac{1}{2})Y \otimes Y - \delta'_2 Y (Y + \frac{1}{2}) \otimes Y \right).$
- $R_{1,0,1,0,0;1,0,1,0,0} = -(\frac{-i\hbar}{2})^2 (X(Y + \frac{1}{2}) \otimes X(Y + \frac{1}{2}) + \delta_1 X(Y + \frac{1}{2}) \otimes Y(Y + \frac{1}{2}))$

Taking the sum of all the above terms, we have that the \hbar^2 component of R is equal to

$$\begin{aligned}
& (\frac{-i\hbar}{2})^2 \left(\frac{1}{2} \delta'_2 (Y + \frac{1}{2})Y \otimes Y + \frac{1}{6} \delta'_2 \otimes (Y + 1)(Y + \frac{1}{2})Y + \frac{1}{2} X^2 \otimes (Y + \frac{1}{2})Y \right. \\
& + \frac{1}{2} (Y + \frac{1}{2})Y \otimes X^2 - X(Y + \frac{1}{2}) \otimes X(Y + \frac{1}{2}) - \delta_1 X(Y + \frac{1}{2}) \otimes Y(Y + \frac{1}{2}) \\
& \left. + \delta_1 (Y + \frac{1}{2})Y \otimes X(Y + \frac{1}{2}) + \frac{1}{2} \delta_1^2 (Y + \frac{1}{2})Y \otimes Y^2 + \frac{1}{4} \delta_1^2 (Y + \frac{1}{2})Y \otimes Y \right) \\
& = (\frac{-i\hbar}{2})^2 \left(\frac{1}{2} S(X)^2 \otimes (Y + \frac{1}{2})Y + S(X)(Y + \frac{1}{2}) \otimes X(Y + \frac{1}{2}) + \frac{1}{2} Y(Y + \frac{1}{2}) \otimes X^2 \right. \\
& \left. + \frac{1}{2} \delta'_2 (Y + \frac{1}{2})Y \otimes Y + \frac{1}{6} \delta'_2 \otimes (Y + 1)(Y + \frac{1}{2})Y + \frac{1}{2} \delta'_2 Y \otimes (Y + \frac{1}{2})Y \right)
\end{aligned}$$

Remark 4.1. The expression of R_2 agrees with the computation in [1]/[Prop. 6.1] up to a term $\frac{1}{12}(\frac{-i\hbar}{2})^2 \delta'_2 \otimes Y$. We notice that $\frac{1}{12}(\frac{-i\hbar}{2})^2 \delta'_2 \otimes Y$ is Hochschild closed. For the purpose of [1]/[Prop. 6.1], a change of a closed Hochschild 2-cochain on \hbar^2 component does not change the answer. This explains the difference.

We remark that when δ'_2 is an inner derivation, we can replace δ'_2 by $[\Omega, \cdot]$ in the expression of R . But it turns out that the above computed \hbar^2 term R_2 does not agree with RC_2 defined by Connes and Moscovici [4], i.e. $R_2|_{\delta'_2[\Omega, \cdot]} - RC_2 =$

$$\Omega Y \otimes Y(2Y + 1) + \Omega Y(2Y + 1) \otimes Y + \frac{1}{3} \Omega \otimes (Y + 1)(2Y + 1)Y - \frac{1}{3} \otimes \Omega(Y + 1)(2Y + 1)Y.$$

We do not know the explicit relation between these two universal deformation formulas $R_2|_{\delta'_2=[\Omega, \cdot]}$ and RC_2 , but only a heuristic and geometric explanation to their difference. The geometric constructions in [1] and in this article are not exactly same. In [1], we used a projective structure to redefine a symplectic connection and a Fedosov connection D' on the Weyl algebra bundle \mathcal{W} , and therefore a symplectic diffeomorphism preserving this new Fedosov connection naturally lifts to an endomorphism of the quantum algebra $\mathcal{W}_{D'}$. In this paper, we

do not change the symplectic connection because of lack of data, but change the quantization process of a symplectic diffeomorphism by introducing sections like U_α, U_β, \dots . Furthermore, we notice that the difference $R_2|_{\delta_2=[\Omega, \cdot]} - RC_2$ is actually a Hochschild coboundary of a 1-Hochschild cochain $-1/3\Omega(Y+1)(2Y+1)Y$. This suggests if we define an isomorphism $I = 1 + 1/3\hbar^2\Omega(Y+1)(2Y+1)Y$ on $C_c^\infty(\mathbb{R} \times \mathbb{R}^+) \rtimes \Gamma[[\hbar]]$, then $I^{-1}(m(R(I(a) \otimes I(b)))) = m + \hbar RC_1 + \hbar^2 RC_2 + o(\hbar^2)$. In general, we expect that if \mathcal{H}_1 acts on \mathcal{A} with a projective structure Ω , there is an isomorphism I on $\mathcal{A}[[\hbar]]$ which can be expressed using elements in $\mathcal{H}_1[[\hbar]]$ and the projective structure Ω such that

$$I^{-1}(m(R(I(a) \otimes I(b)))) = m(RC(a \otimes b)).$$

5. APPENDIX : ASSOCIATIVITY OF THE EHOLZER PRODUCT

In this appendix we study associativity of the Eholzer product, which was used in Connes and Moscovici's approach [4] to obtain the general associativity at the Hopf algebra level. This associativity theorem was first proved by Cohen, Manin, and Zagier in [5]. In the first part of this appendix, we give a new proof of the associativity using the method developed by the second author [8]. In the second part, we study an important combinatorial identity used by Cohen, Manin, and Zagier in [5]. This interesting identity was obtained by Zagier [11], but its complete proof is missing in literature. We prove this identity in the special case corresponding to the Eholzer product.

5.1. Proof of Associativity. First we follow the argument developed in [8] according to which the associativity of the product

$$f * g = \sum_{n=0}^{\infty} [f, g]_n \hbar^n$$

is equivalent to prove the identity (with the notation $X_n = \prod_{i=0}^{n-1} (X + i)$)

$$\sum_{r=0} \binom{n-r}{p} \frac{A_{n-r}(2k+2l+2r, 2m) A_r(2k, 2l)}{(2k+2l+2r)_{n-p-r} (2m)_p (2k)_r} = \sum_{s=0} \binom{n-s}{n-p} \frac{A_{n-s}(2k, 2l+2m+2s) A_s(2l, 2m)}{(2k)_{n-p} (2l+2m+2s)_{p-s} (2m)_s},$$

for $p = 0, 1, \dots, n$ and for

$$A_n(2k, 2l) = \frac{1}{n!} (2k)_n (2l)_n,$$

i.e.

$$\begin{aligned} & \sum_{r=0} \binom{n-r}{p} \frac{\frac{1}{r!} (2k)_r (2l)_r}{(2k)_r} \frac{\frac{1}{(n-r)!} (2k+2l+2r)_{n-r} (2m)_{n-r}}{(2k+2l+2r)_{n-p-r} (2m)_p} \\ (9) \quad &= \sum_{s=0} \binom{n-s}{n-p} \frac{\frac{1}{s!} (2l)_s (2m)_s}{(2m)_s} \frac{\frac{1}{(n-s)!} (2k)_{n-s} (2l+2m+2s)_{n-s}}{(2l+2m+2s)_{p-s} (2k)_{n-p}}. \end{aligned}$$

Our proof is based on manipulation of combinatorial identities. We have, for the left hand side,

$$\begin{aligned} & \sum_{r=0} \binom{n-r}{p} \frac{\frac{1}{r!} (2k)_r (2l)_r}{(2k)_r} \frac{\frac{1}{(n-r)!} (2k+2l+2r)_{n-r} (2m)_{n-r}}{(2k+2l+2r)_{n-p-r} (2m)_p} \\ &= \sum_{r=0} \frac{(n-r)!}{p! (n-r-p)!} \frac{1}{r!} \frac{(2k)_r (2l)_r}{(2k)_r} \frac{1}{(n-r)!} \frac{(2k+2l+2r)_{n-r} (2m)_{n-r}}{(2k+2l+2r)_{n-p-r} (2m)_p} \end{aligned}$$

$$\begin{aligned}
&= \sum_{r=0} \frac{(2l)_r}{r!} \frac{(2k+2l+2r)_{n-r}}{(2k+2l+2r)_{n-p-r} p!} \frac{(2m)_{n-r}}{(2m)_p (n-r-p)!} \\
(10) \quad &= \sum_{r=0} \binom{2l+r-1}{r} \binom{2k+2l+n+r-1}{p} \binom{2m+n-r-1}{n-p-r}.
\end{aligned}$$

Once n, p is fixed, what to be verified is an identity about polynomials in $2k, 2l, 2m$. When $2l$ is a negative integer, we can use the following two combinatorial relations:

$$\binom{X+n-1}{n} = (-1)^n \binom{-X}{n}, \quad n > 0, \quad \text{and} \quad \sum_i \binom{X}{i} \binom{Y}{n-i} = \binom{X+Y}{n},$$

to get

$$\begin{aligned}
\binom{2l+r-1}{r} &= (-1)^r \binom{-2l}{r}, \\
\binom{2k+2l+n+r-1}{p} &= \sum_u \binom{2k+2l+n-1}{p+2l+u} \binom{r}{-2l-u}, \\
\binom{2m+n-r-1}{n-p-r} &= \sum_v \binom{2l+2m+n-1}{n-p+2l+v} \binom{-2l-r}{-2l-r-v}.
\end{aligned}$$

Then (10) becomes

$$\begin{aligned}
&= \sum_{r=0} (-1)^r \binom{-2l}{r} \left[\sum_u \binom{2k+2l+n-1}{p+2l+u} \binom{r}{-2l-u} \right] \left[\sum_v \binom{2l+2m+n-1}{n-p+2l+v} \binom{-2l-r}{-2l-r-v} \right] \\
&= \sum_{u,v} \binom{2k+2l+n-1}{p+2l+u} \binom{2l+2m+n-1}{n-p+2l+v} \left[\sum_r (-1)^r \binom{-2l}{r} \binom{r}{-2l-u} \binom{-2l-r}{-2l-r-v} \right].
\end{aligned}$$

We then simplify the quantity inside the above brackets by

$$\begin{aligned}
&\sum_r (-1)^r \binom{-2l}{r} \binom{r}{-2l-u} \binom{-2l-r}{-2l-r-v} \\
&= \sum_r (-1)^r \frac{(-2l)!}{r!(-2l-r)!} \frac{r!}{(-2l-u)!(-2l-r-v)!} \frac{(-2l-r)!}{(-2l-r-v)!v!} \\
&= \frac{(-2l)!}{(-2l-u)!v!} \sum_r (-1)^r \frac{1}{(r+2l+u)!(-2l-r-v)!} \\
&= \frac{(-2l)!}{(-2l-u)!v!} \frac{1}{(u-v)!} \sum_r (-1)^r \frac{(u-v)!}{(r+2l+u)!(-2l-r-v)!} \\
&= \frac{(-2l)!}{(-2l-u)!v!} \frac{1}{(u-v)!} \sum_r (-1)^r \binom{u-v}{-2l-r-v} \\
&= \frac{(-2l)!}{(-2l-u)!v!} \frac{1}{(u-v)!} (1-1)^{u-v} (-1)^{-2l-v} = \frac{(-2l)!}{(-2l-u)!v!} (-1)^{-2l-v} \delta_{u,v},
\end{aligned}$$

where $\delta_{x,y}$ is the Kronecker symbol ($\delta_{x,y} = 1$ if $x = y$, $= 0$ if not). Finally we get,

$$\begin{aligned}
&\sum_{r=0} \binom{n-r}{p} \frac{1}{r!} (2k)_r (2l)_r \frac{1}{(n-r)!} \frac{(2k+2l+2r)_{n-r} (2m)_{n-r}}{(2k+2l+2r)_{n-p-r} (2m)_p} \\
&= \sum_{u=v=-2l-t} \binom{2k+2l+n-1}{p+2l+u} \binom{2l+2m+n-1}{n-p+2l+v} \frac{(-2l)!}{(-2l-u)!v!} (-1)^{-2l-v}
\end{aligned}$$

$$= \sum_t \binom{2k+2l+n-1}{p-t} \binom{2l+2m+n-1}{n-p-t} (-1)^t \binom{-2l}{t}.$$

On the right hand side of (9), we have

$$\begin{aligned} & \sum_{s=0} \binom{n-s}{n-p} \frac{1}{s!} (2l)_s (2m)_s \frac{1}{(n-s)!} (2k)_{n-s} (2l+2m+2s)_{n-s} \\ &= \sum_{s=0} \frac{(n-s)!}{(n-p)!(p-s)!} \frac{1}{s!} \frac{(2l)_s (2m)_s}{(2m)_s} \frac{1}{(n-s)!} \frac{(2k)_{n-s} (2l+2m+2s)_{n-s}}{(2k)_{n-p} (2l+2m+2s)_{p-s}} \\ &= \sum_{s=0} \binom{2l+s-1}{s} \binom{2k+n-s-1}{p-s} \binom{2l+2m+s+n-1}{n-p}. \end{aligned}$$

By the same method as before, we compute the above quantity as follows,

$$\begin{aligned} &= \sum_{s=0} (-1)^s \binom{-2l}{s} \left[\sum_v \binom{2k+2l+n-1}{p+2l+v} \binom{-2l-s}{-2l-s-v} \right] \left[\sum_u \binom{2l+2m+n-1}{n-p+2l+u} \binom{s}{-2l-u} \right] \\ &= \sum_{u,v} \binom{2k+2l+n-1}{p+2l+v} \binom{2l+2m+n-1}{n-p+2l+u} \left[\sum_{s=0} (-1)^s \binom{-2l}{s} \binom{-2l-s}{-2l-s-v} \binom{s}{-2l-u} \right] \\ &= \sum_{u=v=-2l-t} \binom{2k+2l+n-1}{p+2l+v} \binom{2l+2m+n-1}{n-p+2l+u} \frac{(-2l)!}{(-2l-u)!v!} (-1)^{-2l-v} \\ &= \sum_t \binom{2k+2l+n-1}{p-t} \binom{2l+2m+n-1}{n-p-t} (-1)^t \binom{-2l}{t} \end{aligned}$$

which gives out the same quantity. We obtain then

Proposition 5.1. *The Eholzer product is associative.*

Remark 5.2. *If we want to prove the identification of the coefficients of every $d^u f d^v g d^w h$ in $(f * g) * h$ and $f * (g * h)$, what we proved above is that for every triple of indices (l_1, l_2, l_3) , the following combinatorial identity :*

$$\begin{aligned} & \sum_{r=0}^{l_1} \sum_{s=0}^{l_2} \sum_{t=0}^{l_3} (-1)^{l_1+l_2-s} \binom{E+l_1+s+l_3-t-1}{l_3-t} \binom{E+r+s-1}{s} f \\ & \quad \binom{E+l_2+r+t-1}{t} \binom{E+r+s-1}{r} g \\ & \quad \binom{E+l_1-r+l_2-s+l_3-1}{l_1-r} \binom{E+l_3+l_2-s-1}{l_2-s} h \\ &= \sum_{r=0}^{l_1} \sum_{s=0}^{l_2} \sum_{t=0}^{l_3} (-1)^{l_1+l_2-s} \binom{E+l_1+s+l_3-t-1}{l_3-t} \binom{E+l_1+s-1}{s} f \\ & \quad \binom{E+l_2-s+t-1}{t} \binom{E+l_2+t+r-1}{r} g \\ & \quad \binom{E+l_1-r+l_2-s+l_3-1}{l_1-r} \binom{E+t+l_2-s-1}{l_2-s} h, \end{aligned}$$

where E is the Euler operator.

5.2. Zagier's identity. In this part we turn to the original Cohen, Manin and Zagier's proof [5] of the Eholzer product. Their proof relies on the following combinatorial identity

$$(11) \quad \frac{(-4)^n}{\binom{2x}{n}} \sum_{r+s=n} \frac{\binom{y}{r} \binom{y-a}{r}}{\binom{2y}{r}} \frac{\binom{z}{s} \binom{z+a}{s}}{\binom{2z}{s}} = \sum_{j \geq 0} \binom{n}{2j} \frac{\binom{-\frac{1}{2}}{j} \binom{a-\frac{1}{2}}{j} \binom{-a-\frac{1}{2}}{j}}{\binom{x-\frac{1}{2}}{j} \binom{y-\frac{1}{2}}{j} \binom{z-\frac{1}{2}}{j}}$$

where $n \geq 0$ and the variables a, x, y, z satisfy $x + y + z = n - 1$.

In this part, we give a proof of this identity when $a = \frac{1}{2}$. We start with some transformations. Our aim is to eliminate the binomial coefficients in the denominator of both sides. Using the following identity

$$\frac{1}{\binom{X}{r}} = \frac{\binom{X-r}{n-r}}{\binom{X}{n}} \frac{r!(n-r)!}{n!},$$

we rewrite the left hand side of (11) into the following expression

$$(12) \quad \frac{(-4)^n}{\binom{2x}{n}} \sum_{r=0}^n \frac{\binom{y}{r} \binom{y-a}{r} \binom{2y-r}{n-r}}{\binom{2y}{n}} \frac{\binom{z}{n-r} \binom{z+a}{n-r} \binom{2z-n+r}{r}}{\binom{2z}{n}} \frac{(r!(n-r)!)^2}{(n!)^2}.$$

And using the following identity

$$\binom{2Y}{2j} = \binom{Y-\frac{1}{2}}{j} \binom{Y}{j} \frac{(j!)^2 4^j}{(2j)!},$$

we rewrite the right hand side of (11) into the following expression,

$$(13) \quad \begin{aligned} &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} \binom{-\frac{1}{2}}{j} \binom{a-\frac{1}{2}}{j} \binom{-a-\frac{1}{2}}{j} \frac{\binom{x}{j} 4^j (j!)^2}{\binom{2x}{2j} (2j)!} \frac{\binom{y}{j} 4^j (j!)^2}{\binom{2y}{2j} (2j)!} \frac{\binom{z}{j} 4^j (j!)^2}{\binom{2z}{2j} (2j)!} \\ &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} \binom{-\frac{1}{2}}{j} \binom{a-\frac{1}{2}}{j} \binom{-a-\frac{1}{2}}{j} \left[\frac{4^j (j!)^2 (n-2j)!}{n!} \right]^3 \\ &\quad \frac{\binom{x}{j} \binom{2x-2j}{n-2j}}{\binom{2x}{n}} \frac{\binom{y}{j} \binom{2y-2j}{n-2j}}{\binom{2y}{n}} \frac{\binom{z}{j} \binom{2z-2j}{n-2j}}{\binom{2z}{n}}. \end{aligned}$$

By combining (12) and (13) we can then multiply both sides of (11) by the common denominator $\binom{2x}{n} \binom{2y}{n} \binom{2z}{n}$. We obtain a polynomial $P_n(y, z, a)$ on the left hand side of (11),

$$P_n(y, z, a) := (-4)^n \sum_{r=0}^n \binom{y}{r} \binom{y-a}{r} \binom{2y-r}{n-r} \binom{z}{n-r} \binom{z+a}{n-r} \binom{2z-n+r}{r} (r!(n-r)!)^2,$$

and a polynomial $Q_n(y, z, a)$ on the right hand side of (11), (we replace x by $n - y - z - 1$)

$$Q_n(y, z, a) := \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{((n-2j)!)^2 (j!)^6 2^{6j}}{(2j)!} \binom{-\frac{1}{2}}{j} \binom{a-\frac{1}{2}}{j} \binom{-a-\frac{1}{2}}{j} \binom{n-y-z-1}{j} \binom{2n-2y-2z-2-2j}{n-2j} \binom{y}{j} \binom{2y-2j}{n-2j} \binom{z}{j} \binom{2z-2j}{n-2j}.$$

In summary, identity (11) is equivalent to the identity “ $P_n(y, z, a) = Q_n(y, z, a)$ ” for $n \geq 0$. In order to have the associativity of the Ehlozer product, we shall prove this identity when $a = \frac{1}{2}$. Explicitly, we want to prove the following identity,

$$(14) \quad (-1)^n \sum_{r=0}^n \binom{2y}{2r} \binom{2y-r}{n-r} \binom{2z+1}{2(n-r)} \binom{2z-n+r}{r} \frac{(2r)!(2(n-r))!}{(n!)^2} \\ = \binom{2n-2y-2z-2}{n} \binom{2y}{n} \binom{2z}{n}.$$

5.2.1. *Simplification.* We apply the following identity on the left hand side of (14)

$$\frac{\binom{2y}{2r} \binom{2y-r}{n-r} (2r)!}{n!} = \binom{2y}{n} \binom{2y-r}{r} \frac{r!}{(n-r)!}, \\ \frac{\binom{2z+1}{2(n-r)} \binom{2z-n+r}{r} (2(n-r))!}{n!} = \binom{2z}{n} \left[\binom{2z-n+r}{n-r} + 2 \binom{2z-n+r}{n-r-1} \right] \frac{(n-r)!}{r!}.$$

Taking the quotients on both sides of (14) by $\binom{2y}{n} \binom{2z}{n}$, we have the following equivalent identity,

$$(-1)^n \sum_{r=0}^n \binom{2y-r}{r} \left[\binom{2z-n+r}{n-r} + 2 \binom{2z-n+r}{n-r-1} \right] = \binom{2n-2y-2z-2}{n}.$$

And we can rewrite the above identity as

$$(15) \quad \sum_{r=0}^n \binom{2y-r}{r} \left[\binom{2z-n+r}{n-r} + 2 \binom{2z-n+r}{n-r-1} \right] = \binom{2y+2z-n+1}{n},$$

using $\binom{2n-2y-2z-2}{n} = (-1)^n \binom{2y+2z-n+1}{n}$.

5.2.2. *The sums $S_0(n; A, B)$ and $S(n; X)$.* We consider the sum

$$S_0(n; A, B) = \sum_{k=0}^n \binom{k+A}{n-k} \binom{n-k+B}{k}.$$

We have $S_0(0; A, B) = 1$, and by $\binom{X+1}{n} = \binom{X}{n} + \binom{X}{n-1}$,

$$S_0(n+1; A, B) = \sum_{k=0}^{n+1} \binom{k+A}{n+1-k} \binom{n+1-k+B}{k} \\ = S_0(n; A-1, B+1) + S_0(n+1; A-1, B).$$

In the same way, $S_0(n+1; A, B) = S_0(n; A+1, B-1) + S_0(n+1; A, B-1)$.

On the other hand, we define now

$$S(n; X) = \sum_{p=0}^{\lfloor \frac{n}{2} \rfloor} \binom{X+n-1-2p}{n-2p}.$$

It is clear that $S_0(0; X) = 1$ and $S(n+1; X) = S(n+1; X-1) + S(n; X)$ for the same reason as above.

Another useful relation is that

$$(16) \quad S(n, X-1) + 2S(n-1, X) = \binom{X+n}{n},$$

because

$$\begin{aligned} S(n, X-1) + 2S(n-1, X) &= S(n; X) + S(n-1; X) \\ &= \sum_{p=0}^{\lfloor \frac{n}{2} \rfloor} \binom{X+n-1-2p}{n-2p} + \sum_{p=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{X+n-2-2p}{n-1-2p} \\ &= \sum_{i=0}^n (-1)^i \binom{-X}{i} = (-1)^n \binom{-X-1}{n} = \binom{X+n}{n}. \end{aligned}$$

We prove the following lemma:

Lemma 5.3. $S_0(n; A, B) = S(n; A+B)$.

Proof. We prove the lemma by induction on n . When $n = 0$, we have obviously equality. If the claim is valid for $0, 1, \dots, n$, then by the above two induction relations, we have

$$\begin{aligned} &S_0(n+1; A, B) - S(n+1; A+B) \\ &= (S_0(n+1; A, B-1) + S_0(n; A+1, B-1)) - (S(n+1; A+B-1) + S(n; A+B)) \\ &= (S_0(n+1; A, B-1) - S(n+1; A+B-1)) + (S_0(n; A+1, B-1) - S(n; A+B)) \\ &= S_0(n+1; A, B-1) - S(n+1; A+B-1). \end{aligned}$$

With the same method, we can prove that this difference also equals $S_0(n+1; A-1, B) - S(n+1; A+B-1)$. We hence conclude that the difference $S_0(n+1; A, B) - S(n+1; A+B)$ has the same value for all pairs $(A, B) \in \mathbb{N}^2$. But by definition we can observe

$$S(n+1; 0, 0) - S(n+1; 0) = \begin{cases} 0 - 0 = 0 & \text{if } n \text{ odd,} \\ 1 - 1 = 0 & \text{if } n \text{ even.} \end{cases}$$

We conclude that the identity holds for all n and all pairs $(A, B) \in \mathbb{N}^2$. But as what we want to prove is a polynomial identity in A and B (for fixed n), the identities for all natural numbers implies its correctness for arbitrary (A, B) . \square

5.2.3. Resummation.

Theorem 5.4. *The identity (15) is valid.*

Proof. In fact, by using (16), we have

$$\begin{aligned} &\sum_{r=0}^n \binom{2y-r}{r} \left[\binom{2z-n+r}{n-r} + 2 \binom{2z-n+r}{n-r-1} \right] \\ &= [S_0(n; 2z-n, 2y-n) + 2S_0(n-1; 2z-n, 2y-n+1)] \\ &= [S(n; 2y+2z-2n) + 2S(n-1; 2y+2z-2n+1)] \\ &= \binom{2y+2z-n+1}{n}. \quad \square \end{aligned}$$

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